

Feedback Stability for Dissipative Switched Systems

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Abstract: A method is proposed to infer Lyapunov and asymptotic stability properties for switching systems, under arbitrary continuous-state feedback. Continuous-time systems which are dissipative in the multiple-storage function sense are considered. A partition of the state space, induced by the cross-supply rates and the feedback function, is used to derive conditions for stability. It is argued that the conditions proposed here are more straightforward to check, when compared to those proposed by other approaches in the literature. Some numerical examples are offered to illustrate this point.

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1. INTRODUCTION

Switching systems are dynamic systems, for which the system dynamics switch discretely between different modes of operation or subsystems. Interest in such systems is part of a larger trend, which also includes impulsive systems Haddad et al. (2014), which exhibit discontinuities in the system state, and hybrid systems Goebel et al. (2012), for which switching and impulsive behaviours are combined. The study of these classes of systems has been motivated by the fact that this type of hybrid behaviour is observed in various natural and artificial processes, and the observation that the intricacies of such behaviours could not be captured by pre-existing theories.

Stability is a central problem, both for smooth and for switching systems. Extending the non-switching stability results and techniques (for instance, Lyapunov and Lasalle techniques) to the case of switching systems has been a non-trivial endeavour. Such extensions are explored in Liberzon (2012), where single Lyapunov functions are used, and in Branicky (1998), where multiple Lyapunov functions are applied. Another popular approach includes the introduction of dwell-time conditions: that is, restrictions for the time that the system spends in every mode of operation, as proposed, for example in Shorten et al. (2007). Other problems, usually explored in relation to switching systems, include decidability of various control problems (reachability, controllability etc) Henzinger et al. (1995), and verification Broucke (1999); Navarro-López and Carter (2016). For those problems, systems are represented as hybrid automata, and tools from logics Davoren and Nerode (2000) and reachability theory Lygeros et al. (1999) are adapted to automatically answer questions about the evolution of the trajectories.

Dissipativity was first introduced for continuous-time (non-switching) dynamical systems, in 1972, in Willems (1972). The idea behind it was that the insights gained by

the use of the concept of energy, for example in electric circuits, could be replicated in more abstract dynamic systems. Energy descriptions of dynamic systems are desirable, principally for two reasons. First, they allow for intuitively clear descriptions of system behaviours, and, second, they are versatile, in the sense that a diverse collection of phenomena can be described in terms of energy.

Within the dissipativity framework, the behaviour of the system is described by a supply rate function, which describes the flow of energy in and out of the system, and a storage function, expressing the energy stored in the system at every state. A system is said to be dissipative, if, as it evolves, it dissipates some of the energy that flows into it (which is a form of ‘mild’ behaviour); this idea is made precise in Definition 3 of the next section. For such systems, information about the trajectories can be obtained by studying the energy behaviour.

In the case of switching systems, various extensions of dissipativity theory have been proposed. The majority of those extensions falls within one of two categories. First, single-storage function definitions (Haddad and Chellaboina (2001) and Naldi and Sanfelice (2011)), for which a common supply rate/storage function pair is used to describe the energy behaviour of all the subsystems. Second, multiple-storage function definitions (Zhao and Hill (2008), Zefran et al. (2001), Pogromsky et al. (1998) and Navarro-López and Laila (2013)), for which one pair is used for each subsystem, and the concept of cross-supply rate is introduced to describe the energy effects of the interconnection. Other approaches include the passivity indices, introduced in McCourt and Antsaklis (2010), where switching dissipativity is not defined *per se*, but the dissipativity properties of the subsystems are used to establish results; and the differential inclusions approach proposed in Haddad and Sadikhov (2012), which innovates by using multiple supply rates.

In this work, the multiple-storage function dissipativity framework is used to establish some stability results for switching systems. Similar results already exist in the literature (see the discussion in Section 3), corresponding to the various multiple-storage function definitions of dissipativity. The proposed approach differs from these results, because it deploys the dissipativity framework in a distinct way, and, hence, produces a characterisation of the stability properties that is substantially different. It is argued that this characterisation is often preferable, in the sense that the conditions that it posits require information that is relatively easy to obtain, in comparison to already-existing methods.

The rest of this paper is organised as follows. First, some preliminaries are given, and the main concepts are formally defined. Then, the main results are presented, along with their proofs, and a comparison with already-existing results. Finally, some illustrative examples are given, followed by brief concluding remarks.

2. PRELIMINARIES

For $n, m_o, m_i \in \mathbb{N}$, some natural numbers, let $X = \mathbb{R}^n$, $U = \mathbb{R}^{m_i}$, $Y = \mathbb{R}^{m_o}$ designate the domains of the system state, the input and the output. Take a finite collection of indices $\mathcal{N} = \{1, 2, \dots, N\}$, and a corresponding collection of functions $\mathcal{F} = \{f_1, f_2, \dots, f_N\}$, such that $\forall i \in \mathcal{N}$, $f_i : X \times U \mapsto \mathbb{R}^n$. Take, also, a similar collection of output functions $\mathcal{O} = \{h_1, h_2, \dots, h_N\}$.

In order to describe the switching behaviour, consider $\sigma : \mathbb{R}^+ \mapsto \mathcal{N}$, a piecewise constant, left-continuous function, representing the switching law of the system. Take, also, \mathcal{T} to be the increasing sequence of switching instants $(t_k)_{k \in \mathbb{N}}$. Then, for some $t_k, t_{k+1} \in \mathcal{T}$ and $t_k < \tau_0 \leq t_{k+1} < \tau_1$, it always holds that $\sigma(t_k + 1) = \sigma(\tau)$, but it might be that $\sigma(t_k + 1) \neq \sigma(\tau_1)$.

The systems that will be considered here take the following form:

$$\mathcal{H} \begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t), u(t)), \\ y(t) = h_{\sigma(t)}(x(t), u(t)), \end{cases} \quad (1)$$

with $\sigma(t)$ being the switching law. The notation $f_{\sigma(t)}$ means that $f_{\sigma(t)} = f_i$, when $\sigma(t) = i \in \mathcal{N}$; the same convention holds for $h_{\sigma(t)}$. The implication, then, is that, for \mathcal{H} , the system dynamics between consecutive switches are given by one of the functions in \mathcal{F} . It is common to refer to these functions as the subsystems of \mathcal{H} . It is said, then, that the subsystem i is active, when $\sigma(t) = i$. The elements of \mathcal{F} are assumed to be continuous, and, these elements and the input functions u are assumed to be well-behaved, so that existence and uniqueness issues do not arise.

An additional issue, relevant to the study of switching systems, is the well-known Zeno behaviour, in which the solution of the system cannot be extended beyond some time point, due to the presence of infinite switches. It is assumed, here, that the switching regimes under consideration do not exhibit this kind of behaviour.

An equilibrium point for the system \mathcal{H} is a point $x_* \in X$, for which some $f_i \in \mathcal{F}$ vanishes, for some $u_* \in U$. That is:

Definition 1. (Equilibrium point Khalil (2002)). An equilibrium point for \mathcal{H} is a triplet $(x_*, u_*, i) \in X \times U \times \mathcal{N}$, such that $f_i(x_*, u_*) = 0$. Let \mathcal{E} designate the set (possibly empty) of equilibrium points of \mathcal{H} . ■

Observe that, for every $i \in \mathcal{N}$, multiple equilibrium points might exist.

In this work, some stability properties of the equilibria of switching systems will be examined. The following notion of stability is used.

Definition 2. (Stability Zhao and Hill (2008)). Consider a system \mathcal{H} , starting at $t_0 \geq 0$, with initial state $x_0 = x(t_0) \in X$ under some control $u(t)$ and some switching rule $\sigma(t)$. An equilibrium point $e = (x_*, u_*, i) \in \mathcal{E}$ of \mathcal{H} , under the some control $u(t)$, is said to be:

- **attractive**, iff $\lim_{t \rightarrow \infty} x(t) = x_*$.
- **Lyapunov stable**, iff for each $\epsilon > 0$, there exists $\delta > 0$, such that, if $\|x_* - x(t_0)\| < \delta$, then $\|x_* - x(t)\| < \epsilon$, for all $t \geq t_0$.
- **asymptotically stable**, if it is both attractive and stable. ■

In the next section, some stability conditions will be derived for the subset of the switching systems that are dissipative. To that effect, a multiple-storage function definition of dissipativity, introduced in Zhao and Hill (2008), is used.

Definition 3. (Dissipativity Zhao and Hill (2008)). A system \mathcal{H} is said to be dissipative, with respect to a collection of supply rates $\{s_i\}_{i \in \mathcal{N}}$, and a collection of cross-supply rates $\{r_{ij}\}_{j \in \mathcal{N}/i, i \in \mathcal{N}}$ (/ is used to denote the relative complement), where $\forall i, j \in \mathcal{N}$, $s_i : U \times Y \mapsto \mathbb{R}$, and $r_{ij} : X \times U \times Y \mapsto \mathbb{R}$, with all s_i, r_{ij} locally integrable, if there exists a collection of functions $\{V_i\}_{i \in \mathcal{N}}$, with $V_i : X \mapsto \mathbb{R}^+$, called the storage functions, such that, $\forall t_k, t_{k+1} \in \mathcal{T}$, when $t_k \leq t_k^1 \leq t_k^2 < t_{k+1}$:

- (1) $V_i(x(t_k^1)) - V_i(x(t_k^2)) \leq \int_{t_k^1}^{t_k^2} s_i(u(t), y(t)) dt$, if $\sigma(t_k) = i$.
- (2) $V_i(x(t_k^1)) - V_i(x(t_k^2)) \leq \int_{t_k^1}^{t_k^2} r_{ij}(x(t), u(t), y(t)) dt$, if $\sigma(t_k) = j \neq i$.

For the level lines of V_i , the notation $N_i(\epsilon) = \{x \in X \mid V_i(x) \leq \epsilon\}$ is used.

In essence, the definition posits that a system is dissipative when all its component systems (that is, the members of \mathcal{F}) are dissipative (with respect to some arbitrary supply rates), and it introduces the concept of the cross-supply rate, in order to capture the transfer of energy to some component, caused by the activity of some other component.

In Definition 3, the supply and cross-supply rates satisfy almost identical inequalities and express energy transfers; they appear, then, to be conceptually similar. There is, however, an important distinction which has to be made between them. A supply rate for some subsystem expresses a property of that subsystem, namely, a deep relation between its inputs, its outputs and its state. A cross-supply rate, on the other hand, is an artifact of the connection: that is, of the fact that two subsystems are components of some switching system. Therefore, while the former is

difficult to find, and might not exist, since a subsystem need not have properties of this form, the latter is trivial to find, since the connection is always there (for example, if V_i is differentiable, r_{ij} can always be taken to be $\frac{\partial V_i}{\partial x} f_j$).

3. MAIN RESULT

As was mentioned earlier, numerous multiple-storage function definitions of dissipativity have been proposed. For these definitions, there also exist associated stability results. In Zhao and Hill (2008), in particular, stability is established via a condition that requires that the cross-supply rates are bounded by a function of time. In Zefran et al. (2001) and McCourt and Antsaklis (2010), on the other hand, the condition for stability is that the system energy forms a decreasing sequence on the switching instants of the system. A similar route is taken in Navarro-López and Laila (2013). Then, a common thread to these approaches is that they require some information about the solutions of the system to be available *a priori*.

This information, however, cannot always be obtained, even for simple cases of switching laws. Consider, for example, the law: ‘switch, every time the system crosses from the positive half-space to the negative one with respect to one of the states’. It is clear that for doing this, the switching instants and the corresponding values of the system state are not readily available, and the form of the solutions themselves cannot always be derived. And, conversely, if this wealth of information was available, it is not clear why it would be necessary to use dissipativity (an indirect method), in order to establish stability.

For the purpose of overcoming this deficiency, an alternative approach to establish stability is proposed in this work. The gist of the approach is that the cross-supply rates suggest a partition of the state space, which can be used to deduce whether or not the system is stable. This is because the cross-supply rates can be used to identify areas in which the energy stored in some subsystem is increased, and therefore, areas that should be avoided, if stability is desired.

The idea of introducing state-space partitions has been relatively popular in the literature. In Peleties and De-Carlo (1992), the authors use a partition in a multiple Lyapunov function setting; Skafidas et al. (1999) use a partition induced by a common storage function, in order to draw up conditions for the stability of a class of linear systems, while Zhao and Dimirovski (2004) apply the same approach to non-linear systems. The work by Pogromsky et al. (1998) is more closely related to the results presented here, because it explores the idea of a partition, in conjunction with passivity; that is, the partition is created with respect to the storage functions. Finally, the results in Liu et al. (2010) are important, as they explicitly deals with the issue of zenoness, regarding state-space partitions.

Definition 4. (State-space partition of \mathcal{H}). Take an equilibrium point $e = (x_*, u_*, i) \in \mathcal{E}$ of \mathcal{H} . Consider some $j \in \mathcal{N}$, a function $a : X \mapsto \mathbb{R}$ and a continuous function $v : X \mapsto U$. Define $\mathcal{J} = \mathcal{P}(\mathcal{N}/j)$, where $\mathcal{P}(\cdot)$ denotes the powerset. The partition $\mathcal{K}[e, j, v, a]$ is a collection of sets $\{K_J\}_{J \in \mathcal{J}}$, defined as follows:

$$K_J = \{x \in X \mid r_{ij}(x, v(x), h(x, v(x))) \leq a(x) \forall j \in J\}$$

$$\bigcap \{x \in X \mid r_{ij}(x, v(x), h(x, v(x))) > a(x) \forall j \in \mathcal{N}/J\}$$

where $r_{kk} = s_k$ for every $k \in \mathcal{N}$. ■

The reader should note that the sets defined this way are not overlapping and that $\bigcup_{J \in \mathcal{J}} K_J = X$. While it is clear that the partition exists for any choice for the function a , in the present work, only non-positive and negative definite functions will be considered, corresponding to the idea of negative energy contribution outlined above. Further, it should be clear that, instead of a single a , a collection a_i could be used, one for each subsystem. The resulting partition, which is conceptually identical to the one of Definition 4, will not be explored further to keep the notation tractable.

For this partition, a straightforward Lyapunov argument can be used to draw conclusions on the behaviour of the system. The core idea is that, for any continuous feedback controller v , the partition of Definition 4 induces a rule for the switching, which, when followed, guarantees that the energy of the associated subsystem (that is, the subsystem corresponding to the equilibrium point that generates the partition) decreases. Then, under this rule, stability can be established.

Theorem 5. Consider a system \mathcal{H} and some $e = (x_*, u_*, i) \in \mathcal{E}$. Assume that \mathcal{H} is dissipative according to Definition 3, and that V_i is continuous and positive definite with respect to x_* . Take some initial condition $x(t_0) = x_0$ and the feedback controller $v : X \mapsto U$. Take some non-positive (negative definite) function $a : X \mapsto \mathbb{R}$. Assume that V_i is continuous, positive definite with respect to x_* and radially unbounded¹. For some $k \in \mathcal{N}$, consider the partition $\mathcal{K}[e, k, v, a]$, as given in Definition 4. If the following conditions are held:

- (1) $r_{ik}(x, v(x), h(x, v(x))) \leq a(x)$ for all $x \in X \setminus \{x_*\}$,
- (2) if, for some t , $x(t) \in K_J$, then $\sigma(t) \in J \cup \{k\}$,

then the equilibrium point e is Lyapunov (asymptotically) stable.

Proof. Only the case where $a(x)$ is negative definite is shown; for the non-positive $a(x)$ case, only the first part of the proof is needed.

First, note that V_i is decreasing. To see this, take $\tau_2 \geq \tau_1$. Then, it holds that

$$\begin{aligned} V_i(\tau_2) &\leq V_i(\tau_1) + \int_{\tau_1}^{\tau_2} r_{i\sigma(s)}(x(s), \\ &\quad v(x(s)), h(x(s), v(x(s)))) ds \\ &\leq V_i(\tau_1) + \int_{\tau_1}^{\tau_2} a(x(s)) ds \\ &\leq V_i(\tau_1), \end{aligned} \tag{2}$$

where the second inequality follows from the two conditions of the theorem and the structure of the partition.

A common Lyapunov-style argument is, then, used to establish Lyapunov stability. To see this, note that, for

¹ $V_i(x) \rightarrow \infty$, when $\|x\| \rightarrow \infty$.

every sphere of radius ϵ , $S(\epsilon, x_*)$, one can find $\mu = \min_{x \in B(\epsilon, x_*)} V_i(x)$ (which always exists, as the sphere is compact, and V_i is continuous). Then, any level line $N_i(\mu')$, with $\mu' < \mu$, is wholly contained within $B(\epsilon, x_*)$. Then, for any δ such that $B(\delta, x_*)$ is contained within $N_i(\mu')$, one can be sure (since V_i is decreasing) that, any trajectory with initial state $x(t_0) \in B(\delta, x_*)$ will always remain within $B(\epsilon, x_*)$, hence e is Lyapunov stable.

To show convergence, note that, by assumption, V_i is continuous and lower-bounded by 0, and, as shown above, it is also decreasing. By the monotone convergence theorem Jost (2006), it is deduced that V_i converges to some $\kappa \geq 0$, that is to say, for some initial condition, $\lim_{t \rightarrow \infty} V_i(x(t)) = \kappa$ (note that the conclusion is that there is a κ for every trajectory, not that every trajectory has the same κ).

Assume that, for some initial condition, $\kappa > 0$. Take some $\delta > 0$ (note that the variable name is re-used here), such that $B(\delta, x_*) \subset N_i(\kappa)$. The convergence of V_i implies that the trajectory will never enter $B(\delta, x_*)$. Take $V_i(x_0) = \kappa_0 \geq \kappa$, and $W = N_i(\kappa_0) - B(\delta, x_*)$. Note that W is compact, since $N_i(\kappa_0)$ is closed, by definition, and bounded, by the radial unboundedness of V_i , and $B(\delta, x_*)$ is open. Take $\mu_i(x) = \max_{j \in \mathcal{N}} \{r_{ij}(x, v(x), h(x, v(x)))\}$, which is continuous, since the cross-supply rates are continuous. It follows that $\gamma = \max_{x \in W} \mu_i(x)$, the supremum of a continuous function over a compact set, always exists. Note also that $\gamma < 0$, since it is the maximum over negative values.

Then, by the second part of Inequality (2), it holds that

$$V_i(\tau_2) \leq V_i(\tau_1) + \gamma(\tau_2 - \tau_1), \quad (3)$$

For τ_2 large enough, this implies that $V_i(\tau_2)$ becomes negative. This is a contradiction, caused by assuming that $\kappa > 0$. It holds, then, that, for every initial condition, $\kappa = 0$ and, since $V_i \rightarrow 0$, $\lim_{t \rightarrow \infty} x(t) = x_*$. ■

Some remarks on the function $a(x)$ are due. First, note that the negative definiteness of $a(x)$ is only used to prove convergence to the equilibrium point; indeed, a non-negative $a(x)$ is enough to show Lyapunov stability. This points to the role of $a(x)$ as a (quasi-) design parameter – different choices of $a(x)$ give different, more or less conservative conditions to establish stability. Further, note that, while the same $a(x)$ is used for the partition and the condition on r_{ik} , this is not necessary, and different ones could be used (see the note above on $a(x)$).

The idea behind Theorem 5 is as follows. For some feedback law $v(x)$, and a parameter-function $a(x)$, the state-space is partitioned in different areas, depending on the subset of the subsystems for which the cross-supply rate is negative enough to contribute to the asymptotic behaviour (i.e. energy decrease) of the equilibrium.

The second condition (switching rule) guarantees that the active subsystem always belongs to the subset of (energy-reducing) subsystems. While the condition is referred to as a switching rule, the reader should note that, in reality, it is a family of switching rules – for every K_J , some element of the partition, any (non-Zeno) switching among the elements of J is allowed.

The first condition ensures that this subset of appropriate subsystems is always non-empty – the subsystem k acts

as a fallback option for stability. Evidently, the most reasonable choice for k of this condition is $k = i$, which gives $r_{ii} = s_i$, namely, the supply rate for the subsystem associated to the equilibrium point. Such a choice would correspond to the usual stability condition for dissipative systems, as seen, for example, in Willems (1972) and Zhao and Hill (2008), as well as the intuitive expectation that the energy increases (for a stable subsystem) would be caused rather than compensated by the interconnection. Theorem 5, however, does not require this particular form of the condition.

Theorem 5, then has some appeal, since its conditions assume a form usually encountered in similar results, most notably in Willems (1972). Using the same infrastructure, and considering the same class of systems, one can prove a more powerful result for the asymptotic stability case (the Lyapunov stability case is similar).

Theorem 6. Take the same \mathcal{H} , e , v , a , V_i , as in Theorem 5, with a negative definite. Consider the partition $\mathcal{K}[e, \emptyset, v, a]$, as given in Definition 4. If the following conditions hold:

- (1) $K_\emptyset / \{x_*\} = \emptyset$,
- (2) if, for some t , $x(t) \in K_J$, then $\sigma(t) \in J$,

then the equilibrium point e is asymptotically stable. ■

The proof is omitted, as it is identical to the one of Theorem 5. The reader should note that Theorem 6 is a proper extension of Theorem 5. Indeed, the first condition of the later implies that, for the partition $\mathcal{K}[e, \emptyset, v, a]$, every K_J for which $k \notin J$ is empty, rather than just K_\emptyset . In Theorem 6, instead of taking a particular subsystem to be a fallback option for switching, it is just required that such an option exists everywhere (whatever it might be).

Both of the theorems proposed up to this point are global in character. A local version, however, can be obtained as follows:

Theorem 7. Take some $Q \subset X$ and the same \mathcal{H} , e , v , a , V_i , as in Theorem 5, with a negative definite, and with initial condition $x(t_0) \in Q$. Consider the partition $\mathcal{K}[e, \emptyset, v, a]$. If the following conditions are held:

- (1) $(K_\emptyset / \{x_*\}) \cap Q = \emptyset$,
- (2) if, for some t , $x(t) \in K_J \cap Q$, then $\sigma(t) \in J$,

then, one of the following is true:

- $\exists T > t_0$, such that $x(T) \notin Q$, or
- $\lim_{t \rightarrow \infty} x(t) = x_*$.

Proof. Evidently, either the first condition holds, or the state $x(t)$ remains in Q for all time. In this case, the same argument as in Theorem 5 can be deployed to establish that all the relevant trajectories exhibit asymptotic behaviour, and, therefore, the second case is true. ■

The reader should note that Theorem 7 can be applied multiple times for different (Q, v) pairs; this way, one could conclude stability under a regime where different inputs are used for different areas of the state space. Further, it is clear that the second case of Theorem 7 is relevant only when $x_* \in \bar{Q}$ (the bar stands for the closure of Q), and that the first case is never true, if Q is some level set of V_i . This last remark leads to a corollary of Theorem 7, for region stability.

Corollary 8. Take the same \mathcal{H} , e , v , a , V_i as in Theorem 5. For some $\epsilon > 0$, take $Q^c = N_i(\epsilon)$ (the superscript stands for the complement; then Q is the complement of a level line, where V_i is greater than ϵ) and some initial condition $x(t_0) \in Q$. Consider the partition $\mathcal{K}[e, \emptyset, v, a]$. If the following conditions are held:

- (1) $(K_\emptyset / \{x_*\}) \cap Q = \emptyset$,
- (2) if, for some t , $x(t) \in K_J \cap Q$, then $\sigma(t) \in J$,

then, $\exists T > t_0$, such that $x(T) \notin Q$. ■

It should be clear, from the discussion around Theorem 6, that both Theorem 7 and Corollary 8 can be rephrased in the language of Theorem 5. That is to say, the first condition in both cases can be replaced by a condition on a particular cross-supply rate. The resulting Theorems, however, would be less general than the versions stated here.

Some more general remarks on the results presented here are necessary. First, note that, belonging to the multiple-storage function framework, the approach treats the switching system as a collection of subsystems, for which some information, namely, the dissipativity properties, are known. At the same time, the theorems focus on one of the equilibrium points, and, hence, a single subsystem and a single storage function. From this perspective, the method is similar to the single-storage function approach. This is a ‘best-of-both-worlds’ situation; one can dispense with the tedious effort of finding a storage function/supply rate pair that fits all the subsystems, but, at the same time, avoid time-dependent stability conditions (see the remarks at the beginning of this section).

The idea described here, namely, deriving stability conditions by focusing on a particular subsystem has rarely been used in the literature. An exception is the work in El-Farra et al. (2005), where stability is concluded under an exceptionally restrictive switching rule ($\sigma(t) = j$, for some subsystem j and all t after some specific instant T). A method that is closer to the one presented here is used in Lu and Brown (2010): the authors posit two sets of conditions (one for some subsystem j , and one for the rest of the subsystems), and use them to conclude stability, by adapting the approach of Branicky (1998).

Despite the affinity with the single-storage function approach, note that the V_i of, say, Theorem 5, is not a storage function for the whole system, as it is not necessary that there is a single supply rate for which a dissipation inequality is satisfied. Of course, one could suggest $a(x)$, the parameter function used in the theorems, but this is a red herring; $a(x)$ is only available, because the cross-supply rates are given. Instead, it is known that it satisfies such an inequality for a set-valued supply rate (namely, taking values on the set that comprises of all the supply/cross-supply rates) –the reader should note the similarity with Haddad and Sadikhov (2012).

Another advantage of the approach is that it does not require a common equilibrium for all the subsystems. While V_i is required to have a unique minimum (namely, the x_* of e), there is no need for this point to be a minimum for all the storage functions. Indeed, the rest of the storage functions might have multiple minima, no

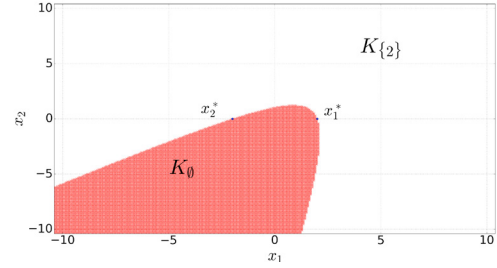


Fig. 1. Visualisation of the partition $\mathcal{K}[e_1, 1, 0, a(x)]$ for system (4) with switching equilibria.

minimum, etc. Further, note that the theorems of this section can be applied separately to every equilibrium point, and they give (potentially) different conditions for stability. The related case of group dissipativity, in which different subsets of the subsystems share dissipativity properties Navarro-López and Laila (2013), can also be treated in a manner that is intuitively clear. Namely, the theorems of this section can be used to obtain conditions of stability, independently for each group of subsystems.

4. NUMERICAL EXAMPLES

The simplest case where the results are applicable is that of the uncontrolled system, where the input remains constant. A simple example illustrating this point is that of a linear, single-input single-output (SISO) system with a switching equilibrium. That is, a 2-mode switching system that takes the following form:

$$\dot{x} = A(x - x_{\sigma(t)}^*) + Bu \quad (4)$$

with $\mathcal{N} = \{1, 2\}$. Take $A = \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, $x_1^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $x_2^* = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$. For such a system, $e_1 = (x_1^*, 0, 1)$ and $e_2 = (x_2^*, 0, 2)$ are two equilibria. By the well-known Kalman-Yakubovich-Popov lemma Khalil (2002), both subsystems can be shown to be passive: that is, $s_i = y^T u$, for some definition of the output $y = C^T x$, for both $i = 1, 2$. In this case, the storage function is

$$V_i = (x - x_i^*)^T P (x - x_i^*), \quad (5)$$

for $i \in \mathcal{N}$, with $P = \begin{bmatrix} 0.833 & 0.167 \\ 0.167 & 0.25 \end{bmatrix}$.

For $u(t) = u_1^* = 0$,

$$r_{12}(x, 0) = 2(x - x_1^*)^T P A (x - x_2^*). \quad (6)$$

Figure 1 shows the partition $\mathcal{K}[e_1, 1, 0, a(x)]$, for $a(x) = 0$, for every $x \in X$. Then, for arbitrary switching in the uncoloured area of Figure 1, and $\sigma(t) = 1$ for the rest, the equilibrium point e_1 is Lyapunov stable, in accordance with Theorem 5.

As a second example, consider the system described by the following equations:

$$\begin{aligned} f_1(x, u) &= \begin{bmatrix} x_2 \\ -x_1^3 - x_2 + u \end{bmatrix}, \\ f_2(x, u) &= \begin{bmatrix} x_2 \\ (1 + x_2^2)(-x_1 - 2x_2 + u) \end{bmatrix}, \\ h_1(x, u) &= h_2(x, u) = x_2. \end{aligned} \quad (7)$$

This system is dissipative with:

$$V_1(x, u) = \frac{x_1^4}{4} + \frac{x_2^2}{2}, \quad (8)$$

$$s_1(u, y) = uy, \quad (9)$$

$$r_{12}(x, u, y) = x_2(1 + x_2^2) \left(\frac{x_1^3}{1 + x_2^2} - x_1 \right) + x_2(1 + x_2^2)u, \quad (10)$$

(the rest of the data is omitted for the sake of brevity).

For this system, $e_1 = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0, 1 \right)$ is an equilibrium point.

Figure 2 shows the partition $\mathcal{K}[e_1, 1, v, a]$ for $v(x) = -0.5x_2$ and $a(x) = -0.01x^T x$. Take an arbitrary set Q , outside K_\emptyset , such as the interior of the red circle, shown in Figure 2. Theorem 7 can be used to guarantee that if the switching happens in accordance to the second condition, then, for any initial condition $x(0) \in Q$, there exists some T , for which $x(T) \notin Q$.

Alternatively, it can be concluded that, for e_1 to be stable, additional modes (with indices $\{3, 4, \dots\}$) should be added to the system, such that, for example, $r_{13}(x, u, y) \leq a(x)$ for some or all $x \in K_\emptyset$. Note that this is a reasonable line of thought, since the whole point of the multiple-storage function outlook is to treat the system as an interconnection of pre-existing components, rather than an immutable whole.

The reader should note that, while, in the examples given here, visual inspection is used to conclude stability, this is not a necessary component of the method. Indeed, the same information could be obtained by examining the supply rates themselves. Take, for instance, the case in which some cross-supply rate is affine with respect to u , that is, it takes the form $r_{ij}(x, u) = \rho_1(x) + \rho_2(x)u$. If, for some area $Q \subseteq \mathbb{R}^n$, it holds that $\mu < \|\rho_2(x)\| \leq \mu'$, then, by taking the feedback function $v(x) = \frac{a(x) - \rho_1(x)}{\rho_2(x)}$ (which is the simplest, although not the best option), one can invoke Theorem 7 to conclude stability. If, additionally, some bound on ρ_1 is known, the stabilisation can be achieved with finite inputs. In a more general setting, one might form the set-valued function $W(x) = \{u \in U \mid r_{ij}(x, u) \leq a(x)\}$, and invoke Michael's Theorem (Michael (1956)), to obtain conditions under which a stabilising feedback function exists.

5. CONCLUSION

This paper has offered a collection of results on the stability of switching systems. It was argued that these results offer additional flexibility, relative to the pre-existing literature, in the sense that the proposed conditions are easier to check, and because they can handle a wider class of dissipative systems (see the discussion on multiple equilibria).

Finally, observe that, while only continuous-time systems are treated here, the exact same approach could be used for discrete-time systems, by rephrasing the definitions appropriately. Impulsive systems can be treated in the same manner (given a definition of dissipativity along the lines of Haddad et al. (2014)), with the partition restricting the reset behaviour (rather than the continuous

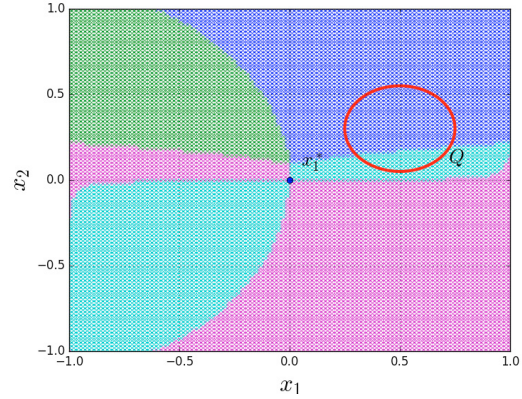


Fig. 2. Visualisation of the partition $\mathcal{K}[e_1, 1, v, a]$ for system (7), and an arbitrary set Q (red). K_\emptyset (magenta), $K_{\{1\}}$ (cyan), $K_{\{2\}}$ (green), $K_{\{1,2\}}$ (blue).

evolution). Proper hybrid systems – like hybrid automata – which exhibit a combination of switching and impulsive behaviours, could also be treated within the same framework, by considering two separate partitions, one for the reset behaviour, and the other for the switching.

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